

# BRANCH GROUPS AND NEW TYPES OF SUBGROUP GROWTH FOR PRO- $p$ GROUPS

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**ABSTRACT.** We show that there are uncountably many different growth types for the subgroup growth of pro- $p$  groups, thus, answering one of the main open problems in the theory of subgroup growth. As a by product we show that a class of pro- $p$  branch groups including the Grigorchuk group and the Gupta-Sidki groups have all subgroup growth type  $n^{\log n}$ .

## 1. INTRODUCTION

Let  $G$  be a group, we denote by  $s_n(G)$  the number of subgroups of  $G$  of index at most  $n$ . Given a function  $f : \mathbb{N} \rightarrow \mathbb{R}$  we say that  $G$  has *subgroup growth of type*  $f(n)$  if there exist  $a, b > 0$  such that  $s_n(G) \leq f(n)^a$  for all  $n$  and  $f(n)^b \leq s_n(G)$  for infinitely many  $n$ . Pyber showed that there is finitely generated profinite group of any 'reasonable' subgroup growth type, see [7]. On the other hand, Shalev showed in [10] that for a pro- $p$  group  $G$  if there exists a constant  $c < \frac{1}{8}$  such that  $s_n(G) \leq n^{c \log n}$  for all  $n$ , then  $G$  has *polynomial subgroup growth*, that is, its subgroup growth is of type  $n$ . Therefore, for pro- $p$  groups there is a gap in the subgroup growth between type  $n$  and type  $n^{\log n}$ .

Pro- $p$  groups of types  $n$ ,  $n^{\log n}$  and  $2^n$  are found in many natural examples. Segal and Shalev [9] constructed metabelian pro- $p$  groups with types  $2^{n^{1/d}}$  for any  $d \in \mathbb{N}$  and Klopsch (unpublished) constructed metabelian pro- $p$  groups with types  $2^{n^{(d-1)/d}}$  for any  $d \in \mathbb{N}$ , see [6, Chapter 9]. No other types were discovered, hence, Lubotzky and Segal posed in [6, Open Problems] the following problems:

### Problem 1.

- (a) *Are there any other gaps in the subgroup growth types of pro- $p$  groups?*
- (b) *What other subgroup growth types occur for pro- $p$  groups?*
- (c) *Is there an uncountable number of subgroup growth types (up to the necessary equivalence) for pro- $p$  groups?*

Lubotzky and Segal were hoping that new examples of subgroup growth types of pro- $p$  groups will arise from studying the subgroup growth of branch groups, see [2] for the definition of branch groups. However, computing the subgroup type of any pro- $p$  branch group illuded the experts for more than a decade. In particular, even the subgroup growth of the well studied Grigorchuk group and the Gupta-Sidki groups remained unknown. In this paper we will show that a particular family of pro- $p$  branch groups, including the Grigorchuk group and the Gupta-Sidki groups, has subgroup growth of type  $n^{\log n}$ . This result is both surprising and interesting as

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previously known examples of pro- $p$  groups with such subgroup growth type are of different nature, that is, either linear and analytic groups or the Nottingham group and its index subgroups. Furthermore, many of these examples are just infinite, that is, their only non-trivial quotients are finite. We therefore suggest the following problem:

**Problem 2.** *Find a just infinite pro- $p$  group with subgroup growth type different from  $n$  and from  $n^{\log n}$ .*

We then use the results above to construct pro- $p$  groups with subgroup growth types of  $n^{(\log n)^k}$  for any positive integer  $k$ . Thus, we give a positive answer to Problem 1(b). Moreover, we are able to construct pro- $p$  groups which obtain uncountable many subgroup growth types slightly faster than all of these growth types and therefore, we give a positive answer to Problem 1(c).

We believe that a variation on our construction can lead to additional growth types and in particular filling the gaps between growth types  $n^{(\log n)^k}$  and  $n^{(\log n)^{(k+1)}}$  for any positive integer  $k$ . However, our current methods of estimating the growth type are not precise enough to enable us to compute the growth in this construction. We hope that further study will enable us to do that.

**Conjecture 3.** *There is no gap in the subgroup growth types of pro- $p$  groups between  $n^{(\log n)^k}$  and  $n^{(\log n)^{(k+1)}}$  for all  $k \in \mathbb{N}$ .*

## 2. RESULTS

A group  $G$  is called *self-replicating*, if there exists an integer  $k$  and a normal subgroup  $G_1 \leq G$  of finite index, such that  $G_1 \cong G^k$  the cartesian product of  $k$  copies of  $G$ . By iterating this process we obtain  $G_n \cong G_{n-1}^k \cong G^{k^n}$  the  $n$ -th *principal congruence subgroup* of  $G$ . We say that  $U < G$  is a *congruence subgroup*, if  $U$  contains a principal congruence subgroup. If  $G$  contains a self-replicating normal subgroup of finite index and every finite index subgroup of it is a congruence subgroup, then we say that  $G$  has the *congruence subgroup property*. (We comment that this definition is slightly different than the standard definition for the congruence subgroup property for groups acting on trees.)

For a pro- $p$ -group  $G$  denote by  $d(G)$  the minimal number of generators of  $G$ , and by  $d_G(m)$  the maximum of  $d(U)$ , as  $U$  ranges over all subgroups of index  $p^m$ . In case it is clear from the context to which  $G$  we refer, we will omit the subscript  $G$ . We say that  $G$  has *logarithmic rank gradient*, if  $\alpha(G) = \lim_{m \rightarrow \infty} \frac{d(m)}{m}$  exists, finite and positive. Notice that if  $(G : U) = p^k$ , then  $d(G) - k \leq d(U) \leq p^k(d(G) - 1) + 1$ . Combining [6, Proposition 1.6.2] and [6, Lemma 4.2.1] it follows that

**Proposition 1.** *Let  $G$  be a pro- $p$  group.*

(1) *If  $G$  has logarithmic rank gradient  $\alpha$ , then*

$$p^{\frac{\alpha^2}{4(\alpha+1)}m^2 + o(m^2)} \leq s_{p^m}(G) \leq p^{\frac{\alpha}{2}m^2 + o(m^2)}.$$

(2) *If  $H$  is commensurable with  $G$ , then  $H$  also has logarithmic rank gradient  $\alpha$ .*

(3) *If  $\mu \leq d(m - \mu)$ , then*

$$p^{\mu(d(m-\mu)-\mu)} \leq s_{p^m}(G) \leq p^{\sum_{\mu=1}^{m-1} d(\mu)}$$

Our first result is the following.

**Theorem 2.** *Let  $G$  be a finitely generated, self-replicating pro- $p$  group that has the congruence subgroup property, then  $G$  has subgroup growth of type  $n^{\log n}$ . More precisely, suppose  $G$  contains a principal congruence subgroup  $N$ , such that  $N \cong G^k$ ,  $|G/N| = p^\ell$ , and  $N$  is contained in the Frattini subgroup of  $G$ . Put  $d = \max d(H)$ , where  $H$  runs over all subgroups of the finite group  $G/N$ . Then we have that*

(1)

$$\frac{(k-1)d(G)}{\ell} \leq \alpha(G) \leq (k-1)d(G) + d;$$

(2)

$$\alpha(G) = \lim_{m \rightarrow \infty} \frac{d(m)}{m} = \sup_{m \geq 0} \frac{d(m)}{m + \frac{\ell}{k-1}}.$$

From this we immediately obtain the following.

**Corollary 3.** *If  $G$  is the Grigorchuk group, then*

$$2^{\frac{9}{40}m^2 + o(m^2)} \leq s_{2^m}(G) \leq 2^{6m^2 + o(m^2)}.$$

*If  $G$  is the Gupta-Sidki  $p$ -group with  $p \geq 3$ , then*

$$p^{\frac{1}{8}m^2 + o(m^2)} \leq s_{p^m}(G) \leq p^{\frac{3p^2 - 4p + 1}{2}m^2 + o(m^2)}.$$

*Proof.* Suppose  $G$  is the Grigorchuk group. Since  $G$  is a 2-group its profinite completion is the same as its pro-2 completion. It follows from [2, Proposition 8] that  $G$  has a self-replicating subgroup  $K$  of index 16, which is 3-generated,  $K_1$  the first principal congruence subgroup is of index 4 with  $K_1 \cong K \times K$ , and  $K/K_1 \cong C_4$ . Also, from [2, Proposition 10]  $G$  has the congruence subgroup property. Thus, we can assume that  $G$  is a pro-2 group with the congruence subgroup property. The Frattini subgroup of  $K$  contains  $N = K_2 \cong K_1 \times K_1 \cong K^4$  of index  $4^3 = 2^6$  in  $K$ . Hence, in the theorem we have  $\ell = 6$ ,  $k = 4$  and  $d = 3$ , thus,  $\frac{3}{2} \leq \alpha(K) \leq 12$ . From Proposition 1 we obtain that the same inequality holds for  $\alpha(G)$ , and our claim follows.

Suppose  $G$  is the Gupta-Sidki group  $p$ -group. Since  $G$  is a  $p$ -group its profinite completion is the same as its pro- $p$  completion. The lower bound for the Gupta-Sidki groups is just the lower bound for non- $p$ -adic analytic pro- $p$  groups, established by Shalev [10]. Note that the following properties of the Gupta-Sidki groups were established by Garrido and Wilson in [4]. For the upper bound we use the fact that  $K$ , the commutator of subgroup of  $G$ , is self-replicating, and we have that  $\Phi(K) \geq K_2$ ,  $K_1 \cong K^p$ ,  $(K : K_2) = p^{p^2-1}$ , and  $d(K) = p - 1$ . Moreover,  $K$  has the congruence subgroup property, and  $d = p(p - 1)$ , the maximum being attained by the subgroup  $K_1/K_2$  of  $K/K_2$ . Hence, by Theorem 2 we have  $\alpha(G) \leq (k - 1)d(K) + d = 3p^2 - 4p + 1$ , and our claim follows from Proposition 1.  $\square$

Our second result shows that there is an infinitude of growth types below  $2^{n^\epsilon}$  for all  $\epsilon > 0$ , which are realized by pro- $p$  groups.

**Theorem 4.** *For every integer  $k$  and prime number  $p$  there exists a pro- $p$  group  $H_k$  with subgroup growth type  $n^{(\log n)^k}$ .*

Our construction depends on the fact that the Grigorchuk group and the Gupta-Sidki groups act on the  $p$ -adic tree as transitive as pro- $p$  groups can, that is, if  $U$  is a subgroup of finite index, then the number of  $U$ -orbits into which the tree decomposes grows only logarithmically with the index of  $U$ .

Furthermore, we can let  $k$  tend to infinity with  $n$ .

**Theorem 5.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function, such that  $f(p^n) \leq f(n) + 1$  holds true for all  $n$ . Then there exists a pro- $p$  group of strict subgroup growth type  $n^{(\log n)^{f(n) + \mathcal{O}(1)}}$ . In particular, there are uncountably many non-equivalent subgroup growth types for pro- $p$  groups.*

From this result we not only obtain the existence of continuum many growth types, but also that the growth types of pro- $p$  groups do not form a linear order. More precisely, we get the following.

**Corollary 6.** *The lattice  $(\mathbb{R}, <)^{\mathbb{N}}$  can be embedded into the lattice of growth types of pro- $p$  groups.*

All these growth types are extremely slow. In fact, the assumption on  $f$  implies  $f(n) \leq \log^* n$ , where  $\log^*$  is the inverse tower function, that is,  $\log^* n$  is the least integer  $k$ , such that

$$p^{p^{\cdots p}} \geq n,$$

where the tower has height  $k$ .

In fact we can prove the existence of uncountably many subgroup growth types between  $n^{(\log n)^{\log^* n}}$  and  $n^{(\log n)^{\log \log n}}$ , however, our estimates (see Lemma 24 below) are not sharp enough to describe them in a decent way. The main problem is that even if a group has the congruence subgroup property a subgroup of rather small index can have quite large congruence level.

**Problem 4.** *Let  $G$  be the Grigorchuk group or a Gupta-Sidki group, or more generally a self replicating branch group. Is there a constant  $C$ , such that for every finite index subgroup  $U$  there exist nodes  $x_1, \dots, x_k$  in the tree, such that the stabilizer  $G_{x_1 \dots x_k}$  is contained in  $U$ , and  $(G : G_{x_1 \dots x_k}) \leq (G : U)^k$ ?*

### 3. PROOF OF THEOREM 2

The following is almost trivial. However, we include the proof since direct products are quite often not as trivial as one might believe.

**Lemma 7.** *Let  $G$  and  $H$  be pro- $p$ -groups,  $m$  an integer, and assume that there is a constant  $C$ , such that for all subgroups  $U_1 < G$ ,  $U_2 < H$  with  $(G : U_1) = p^k$ ,  $(H : U_2) = p^\ell$  with  $k, \ell \leq m$  we have  $d(U_1) \leq Ck + d(G)$ ,  $d(U_2) \leq C\ell + d(H)$ . Then for all subgroups  $U < G \times H$  with index  $p^m$  we have  $d(U) \leq Cm + d(G) + d(H)$ .*

*Proof.* Let  $U$  be a subgroup of  $G \times H$ . Let  $\pi : U \rightarrow H$  be the canonical projection. Then consider  $U_1 = U \cap G$ ,  $U_2 = \text{im } \pi$ . Take generators  $g_1, \dots, g_r$  of  $U_1$ , and generators  $h_1, \dots, h_s$  of  $U_2$ . For each  $h_i$  choose a pre-image  $\tilde{h}_i$  under  $\pi$ . Then  $g_1, \dots, g_r, \tilde{h}_1, \dots, \tilde{h}_s$  are contained in  $U$ , and generate a subgroup  $\tilde{U}$  of  $U$ . We have  $\tilde{U} \cap G = U \cap G$ , and  $\pi(\tilde{U}) = \pi(U)$ , thus,  $(G \times H : \tilde{U}) = (G \times H : U)$ , therefore,  $U = \tilde{U}$ , and  $d(U) \leq r + s$ .

Since  $(G \times H : U) = (G : U_1)(H : U_2)$  by our assumption we get that  $r + s \leq \log(G : U_1) + d(G) + \log(H : U_2) + d(H) = \log(G \times H : U) + d(G) + d(H)$ ,

and our claim follows.  $\square$

**Lemma 8.** *Let  $G$  be a finitely generated self-replicating pro- $p$ -group that has the congruence subgroup property. Define  $N$ ,  $k$  and  $d$  as in Theorem 2. Then a subgroup of index  $p^m$  has at most  $Cm + d(G)$  generators, where  $C = (k-1)d(G) + d$ .*

*Proof.* We prove our assertion by induction on  $m$ . If  $m = 0$ , then  $U = G$ , and our claim is trivial. Now let  $U$  be an open subgroup of index  $p^m$ . If  $UN/N = G/N$ , then in particular  $U\Phi(G)/\Phi(G) = G/\Phi(G)$ , and we conclude that  $U = G$  again. Henceforth we assume that  $UN/N < G/N$ . Put  $V = U \cap N$ . Then

$$(G : U) = (G : UN)(UN : U) = (G/N : UN/N)(N : V),$$

and thus,  $(N : V) < (G : U)$ . As  $V \leq N \cong G^k$  we can use Lemma 7 and the induction hypothesis to obtain that  $d(V) \leq C(m-1) + kd(G)$ . Thus,

$$d(U) \leq d(V) + d(UN/N) \leq d(V) + d.$$

We deduce that

$$\begin{aligned} d(U) &\leq C(m-1) + kd(G) + d = \\ &((k-1)d(G) + d)(m-1) + ((k-1)d(G) + d) + d(G) = Cm + d(G), \end{aligned}$$

and our claim is proven.  $\square$

We now turn to the proof of Theorem 2 (2). The inequality  $\lim \frac{d(m)}{m} \leq \sup \frac{d(m)}{m+c}$  holds for any sequence  $d(m)$  and for any real number  $c$ . For the reverse inequality we start by showing that for any  $r \geq 0$  we can find in  $G$  a subgroup of index  $p^{\ell r}$  which is isomorphic to  $G^{(k-1)r+1}$ . This is done by induction on  $r$ . For  $r = 0$  we take  $G$  itself. Suppose we know it for  $r$ , we will prove it for  $r+1$ . Take one of the component in the subgroup isomorphic to  $G^{(k-1)r+1}$  and replace it by  $N$ . The index of the new subgroup increases by  $p^\ell$  so it is  $p^{\ell(r+1)}$  while the number of components increase by  $k-1$ , so we obtain that the new subgroup is isomorphic to  $G^{(k-1)(r+1)+1}$  as required.

Let  $U < G$  be a subgroup of index  $p^m$  with  $d(U) = d(m)$ . By taking a subgroup isomorphic to  $U$  in each component of  $G^{(k-1)r+1}$  we have that  $G$  contains a subgroup  $V \cong U^{(k-1)r+1}$  of index  $p^{\ell r}(G : U)^{(k-1)r+1} = p^{\ell r}(p^m)^{(k-1)r+1} = p^{(\ell+m(k-1))r+m}$  with  $d(V) = ((k-1)r+1)d(U) = ((k-1)r+1)d(m)$ .

For an integer  $n$  take  $r$  such that  $(\ell+m(k-1))r+m \leq n < (\ell+m(k-1))(r+1)+m$ , that is,  $r = \left\lfloor \frac{n-m}{\ell+m(k-1)} \right\rfloor$ , pick a subgroup  $V$  as described in the last paragraph, and choose  $H < V$  with  $(V : H) = p^{n-(\ell+m(k-1))r-m}$ . Then  $(G : H) = p^n$ , and  $(V : H) < p^{\ell+m(k-1)}$ , thus,  $d(H) \geq d(V) - \ell - m(k-1) = ((k-1)r+1)d(m) - \ell - m(k-1)$ . We conclude that for all  $n$  we have

$$\begin{aligned} d(n) &\geq d(H) \geq ((k-1)r+1)d(m) - \ell - m(k-1) = \frac{(\ell+m(k-1))r - \ell r}{m} d(m) + \mathcal{O}(1) \\ &\geq \frac{(n-m) - \ell - m(k-1) - \ell r}{m} d(m) + \mathcal{O}(1) = \frac{n - \ell r}{m} d(m) + \mathcal{O}(1). \end{aligned}$$

Now dividing by  $n$  we obtain

$$\begin{aligned} \frac{d(n)}{n} &\geq \frac{n - \ell r}{nm} d(m) + \mathcal{O}\left(\frac{1}{n}\right) = \frac{d(m)}{m} \left(1 - \frac{\ell}{\ell + m(k-1)}\right) + \mathcal{O}\left(\frac{1}{n}\right) = \\ &\quad \frac{d(m)}{m} \frac{m(k-1)}{\ell + m(k-1)} + \mathcal{O}\left(\frac{1}{n}\right) = \frac{d(m)}{\frac{\ell}{k-1} + m} + \mathcal{O}\left(\frac{1}{n}\right), \end{aligned}$$

and our claim follows.

Finally note that the lower bound in Theorem 2 (1) follows from Theorem 2 (2) since

$$\sup_{m \geq 0} \frac{d(m)}{m + (\ell/(k-1))} \geq \frac{d(0)}{\frac{\ell}{k-1}} = \frac{(k-1)d(G)}{\ell}.$$

#### 4. PROOF OF THEOREM 4

For  $p = 2$  let  $G$  be the first Grigorchuk group and let  $K$  be the self-replicating subgroup of index 16 in it, while for  $p > 2$  let  $G$  be the Gupta-Sidki  $p$ -group and let  $K$  be the self-replicating subgroup of it of index  $p^2$  (see [3]). Let us emphasize that in this section we take  $G$  and  $K$  to be the discrete  $p$ -groups rather than their pro- $p$  completions. Let  $T$  be the  $p$ -adic tree,  $G$  acts naturally on  $T$  and so does  $K$ . Let  $T_n$  be the  $n$ -level of the tree and let  $St(n) = St_G(T_n)$  be the stabilizer of  $T_n$ , where the root is considered as  $T_0$ , thus  $St(0) = G$ . Note that  $St(1)$  contains  $K^p$  geometrically, that is, that is the embedding induced by considering each child tree of the first level as isomorphic to  $T$ . For  $p = 2$  this was proven by Grigorchuk [2, Proposition 8], for  $p = 3$  by Sidki [11], and for  $p > 3$  by Garrido [3, Proposition 2.2].

Clearly  $K$  acts on the set of infinite paths in  $T$ . Pick one such path  $x$ , and let  $X = \{x^g : g \in K\}$  be the orbit of  $x$  under  $K$ . By taking componentwise action we obtain a transitive permutation group  $(K^k, X^k)$ ,  $k \geq 2$ . Let  $H_k$  be the restricted wreath product  $(K^k, X^k) \wr \mathbb{F}_p$ . Then  $H_k$  is a finitely generated  $p$ -group. Let  $B$  be the base group of the wreath product and  $\pi : H_k \rightarrow K^k$  be the projection onto the active group.

**Lemma 9.** *Let  $G$  be a residually finite group,  $X$  a set on which  $G$  acts transitively. Suppose that for all  $x, y \in X$ , there exists a finite index subgroup  $U$  such that  $x$  and  $y$  are not in the same  $U$ -orbit. Then the wreath product  $G \wr \mathbb{F}_p$  given by the action is residually finite.*

*Proof.* Let  $(g, f)$  be a non-trivial element of the wreath product, where  $g \in G$  and  $f : X \rightarrow \mathbb{F}_p$  is a function with finite support. If  $g \neq 1$ , then there exists a subgroup  $U < G$  of finite index not containing  $g$ , and the pre-image of  $U$  under the canonical projection is a finite index subgroup not containing  $(g, f)$ . Now suppose that  $g = 1$ ,  $f$  does not vanish identically, and write  $\text{supp } f = \{x_1, \dots, x_n\} \neq \emptyset$ . For  $1 \leq i < j \leq n$  choose a finite index subgroup  $U_{ij}$ , such that  $x_i$  and  $x_j$  are not in the same  $U_{ij}$ -orbit. Then  $U = \bigcap U_{ij}$  is a finite index subgroup, which acts on  $X$  with finitely many orbits  $X_1, \dots, X_N$ . Then  $M = \{m : X \rightarrow \mathbb{F}_p : |\text{supp } m| < \infty, \forall i \leq N : \sum_{x \in X_i} m(x) = 0\}$  is  $U$ -invariant and of finite index in  $\mathbb{F}_p^X$ , hence  $\{(u, m) \in G \wr \mathbb{F}_p : u \in U, m \in M\}$  is a finite index subgroup of  $G$ , which does not contain  $(g, f)$ , because each  $X_i$  contains at most one point on which  $f$  does not vanish, and  $f$  does not vanish identically.  $\square$

**Lemma 10.** *Let  $G$  be a  $p$ -group acting faithfully on  $T$ . Then  $(G : St(\ell)) \leq p^{p^\ell}$ .*

*Proof.* There is an injective homomorphism  $G/St(\ell) \rightarrow S_{p^\ell}$ . The image of this homomorphism is a  $p$ -subgroup, hence a subgroup of the  $p$ -Sylow subgroup of  $S_{p^\ell}$ .

The latter has order  $p^{\frac{p^\ell-1}{p-1}}$ , and our claim follows.  $\square$

**Lemma 11.** *Fix  $\ell$  an integer. Let  $U < H_k$  be the preimage of  $C = St(\ell)^k \leq K^k$  under  $\pi$ , that is  $C \wr \mathbb{F}_p$ . Then  $(H_k : U) \leq p^{kp^\ell}$  and there exists  $c > 0$  depending only on  $p$  and  $k$  such that  $U$  maps onto  $V$  an  $\mathbb{F}_p$  vector space with  $\dim_{\mathbb{F}_p} V \geq cp^{k\ell}$ . Therefore,  $d(U) \geq cp^{k\ell}$ . Furthermore,  $H_k$  is residually finite.*

*Proof.* Recall that  $G$  acts transitively on  $T_\ell$ . As  $K$  is a finite index subgroup in  $G$ , the number of orbits of  $K$  acting on  $T_\ell$  is bounded by  $(G : K)$ , which is independent from  $\ell$ . Moreover,  $K$  is normal, so all orbits of  $K$  on  $T_\ell$  have equal size, which is at least  $\frac{p^\ell}{(G:K)}$ . The number of orbits of  $St(\ell)$  acting on  $T_\ell$  is the size of such  $K$ -orbit.

We deduce that the action of  $St(\ell)$  on  $T_\ell$  and therefore on  $X$  has at least  $\frac{p^\ell}{(G:K)}$  orbits. Thus,  $C$  acting on  $X^k$  has at least  $\left(\frac{p^\ell}{(G:K)}\right)^k = cp^{k\ell}$  orbits, where  $c > 0$  depends only on  $p$  and  $k$ .

Let  $\mathcal{O}$  be the set of orbits of  $C$  acting on  $X^k$  and let  $O \in \mathcal{O}$ . We define a map  $\varphi_O : U \rightarrow \mathbb{F}_p$  as follows. Write an element of  $U$  as  $(g, f)$ , where  $g \in C$  and  $f : X^k \rightarrow \mathbb{F}_p$  has finite support. Then we define  $\varphi_O((g, f)) : U \rightarrow \mathbb{F}_p$  by  $\sum_{x \in O} f(x)$ . This map is a homomorphism since

$$\begin{aligned} \varphi_O((g, f)(\tilde{g}, \tilde{f})) &= \varphi_O((g\tilde{g}, f\tilde{g} + \tilde{f})) = \sum_{x \in O} f\tilde{g}(x) + \sum_{x \in O} \tilde{f}(x) \\ &= \sum_{x \in O} f(x\tilde{g}) + \sum_{x \in O} \tilde{f}(x) = \sum_{x \in O} f(x) + \sum_{x \in O} \tilde{f}(x) = \varphi_O((g, f)) + \varphi_O((\tilde{g}, \tilde{f})), \end{aligned}$$

Let  $V = \mathbb{F}_p^{|\mathcal{O}|}$ , we define  $\varphi : U \rightarrow V$  by  $\varphi((g, f)) = (\varphi_O((g, f)))_{O \in \mathcal{O}}$ . For each  $O \in \mathcal{O}$  we fix  $x_O \in O$ . Given  $(a_O) \in \mathbb{F}_p^{|\mathcal{O}|}$  we define  $f : X^k \rightarrow \mathbb{F}_p$  by  $f(x_O) = a_O$  for all  $O \in \mathcal{O}$  and otherwise  $f(x) = 0$ . Then  $\varphi(1, f) = (a_O)$ . Thus,  $\varphi$  is a surjective homomorphism.

The subgroup  $U$  is normal in  $H_k$ , and  $H_k/U \cong (K/St(\ell))^k$ . From Lemma 10 we obtain  $(H_k : U) = |K/St(\ell)|^k < p^{kp^\ell}$ .

Finally the fact that  $H_k$  is residually finite follows from Lemma 9 and the fact that any two paths can be separated by a principal congruence subgroup.  $\square$

One can give more precise bounds for the index of  $U$ . Doing so is essentially equivalent to determining the Hausdorff dimension of  $G$ , which was done for  $p = 2$  by Grigorchuk [2, Section 5], and for  $p \geq 3$  in vast generality by Reizabal [8].

From this result together with the lower bound of Proposition 1 (3) we obtain the lower bound implied by Theorem 4.

**Corollary 12.** *The growth type of  $H_k$  is at least  $n^{(\log n)^k}$ .*

To prove the upper bound we notice that as  $H_k$  is a  $p$ -group its subgroup growth is the same as its pro- $p$  completion. Also since  $H_k$  is residually finite the index of a subgroup of the pro- $p$  completion is the same of the index of its preimage. Moreover, its minimal number of generators is bounded from above by the minimal number of generators of its preimage. Thus, to bound from above the subgroup growth of  $H_k$  it suffices to consider the minimal number of generators of subgroups

of finite index. This falls into two parts: First we show that Lemma 11 is close to best possible, that is, subgroups of  $K^k$  with index  $p^m$  act with  $\mathcal{O}(m^k)$  orbits, which implies that subgroups of  $H_k$  containing  $B$  need  $\mathcal{O}(m^k)$  generators. In the second step we show that a subgroup  $U$  which does not contain  $B$  does not need many more generators than  $UB$ .

**Lemma 13.** *Let  $(S, X)$ ,  $(T, Y)$  be two permutation groups. Let  $U$  be a subgroup of  $S \times T$ . Suppose that  $U \cap S$  has  $m$  orbits on  $X$ , and  $US/S < T$  has  $n$  orbits on  $Y$ . Then  $U$  has at most  $mn$  orbits on  $X \times Y$ .*

*Proof.* Let  $x_1, \dots, x_m$  be representatives of  $X/(U \cap S)$ , and  $y_1, \dots, y_n$  be representatives of  $Y/(US/S)$ . Then every element  $(x, y) \in X \times Y$  is equivalent to an element of the form  $(z, y_j)$ ,  $1 \leq j \leq n$ . Take an element  $u \in U \cap S$  such that  $z^u = x_i$  for some  $1 \leq i \leq m$ . By applying it to  $(z, y_j)$  we obtain that  $(x, y)$  is equivalent  $(x_i, y_j)$ .  $\square$

**Lemma 14.** *Let  $U$  be a subgroup of  $K^k$  of index  $p^m$ . Then  $U$  has at most  $\mathcal{O}(m^k)$  orbits on  $X^k$ .*

*Proof.* We prove our claim by induction on  $k$ . For  $k = 1$  we prove by induction on  $m$  the stronger claim that  $U$  has at most  $(p^2 - 1)m + 1$  orbits on  $X$ .

If  $m = 0$ , there is nothing to show, henceforth  $m \geq 1$ . Grigorchuk [2, Proposition 9] showed that in the case  $p = 2$  we have that  $\Phi(K)$  contains  $St(2)$ , for  $p \geq 3$  the same was proven by Garrido [3, Proposition 2.6]. Since  $U$  is a proper subgroup of  $K$ , we have  $U\Phi(K) \neq K$  and hence  $USt(2) \neq K$ . Thus, we have

$$(St(2) : U \cap St(2)) = (USt(2) : U) < (K : U).$$

As  $St(2)$  contains  $K^{p^2}$  geometrically, we have that

$$(K^{p^2} : U \cap K^{p^2}) \leq (St(2) : U \cap St(2)) < (K : U).$$

Because  $U \cap K^{p^2}$  is a subgroup of  $U$  the number of orbits of  $U$  on  $X$  is bounded from above by the number of orbits of  $U \cap K^{p^2}$ .

Let  $X_i$  be the elements of  $X$  such that the second node in such an element is the  $i$ -th node in the second level. In other words,  $X_i$  are the intersection of  $X$  with the second level  $i$ -th subtree. Then  $X$  is a disjoint union of the  $X_i$  and each  $X_i$  is invariant under  $U \cap K^{p^2}$  because  $U \cap K^{p^2} \leq St(2)$ . Thus, the number of orbits of  $U \cap K^{p^2}$  acting on  $X$  is the sum of the number of orbits of  $U \cap K^{p^2}$  acting on  $X_i$ . Write  $K^{p^2} = K_1 \times \dots \times K_{p^2}$ , where  $K_i \cong K$ . Let  $\tilde{U}_i$  be the projection of  $U \cap K^{p^2}$  onto  $K_i$ . The action of  $U \cap K^{p^2}$  on  $X_i$  factors through  $\tilde{U}_i$  so the number of orbits of  $U \cap K^{p^2}$  acting on  $X_i$  is the number of orbits of  $\tilde{U}_i$  acting on each  $X_i$ .

For  $1 \leq i \leq p^2$  let  $U_i$  be the projection of  $U \cap (K_1 \times \dots \times K_i)$  onto  $K_i$ . Then  $U_i$  is a subgroup of  $\tilde{U}_i$ , so the number of orbits of  $\tilde{U}_i$  acting on  $X_i$  is bounded above by the number of orbits of  $U_i$  acting on  $X_i$ .

Using induction on  $r$  it is easy to prove that  $(K_1 \times \dots \times K_r : U \cap K_1 \times \dots \times K_r) = \prod_{i=1}^r (K_i : U_i)$ . We deduce that

$$\prod_{i=1}^{p^2} (K_i : U_i) = (K^{p^2} : U \cap K^{p^2}) = (UK^{p^2} : U) \leq (U\Phi(K) : U) < (K : U),$$

in particular, each single factor on the left is strictly smaller than  $p^m$ . Viewing  $U_i$  as a subgroup of  $K_i$  acting on  $i$ -th subtree we can apply our induction



hypothesis to find that the number of orbits of  $U \cap K^{p^2}$  acting on  $X$  is at most

$$\begin{aligned} \sum_{i=1}^{p^2} ((p^2 - 1) \log(K_i : U_i) + 1) &= (p^2 - 1) \log(K^{p^2} : U \cap K^{p^2}) + p^2 \\ &\leq (p^2 - 1)(m - 1) + p^2 = (p^2 - 1)m + 1, \end{aligned}$$

the case  $k = 1$  is complete.

Now suppose that  $k \geq 2$ , and our claim is already shown for  $k - 1$ . Write  $K^k = K_1 \times \cdots \times K_k$ . Assume that  $(UK_1 : U) = (K_1 : U \cap K_1) = p^\ell$ . Then from the induction hypothesis for  $k = 1$  we deduce that  $U \cap K_1$  acts with  $\mathcal{O}(\ell)$  orbits on  $X$ . Notice that  $UK_1/K_1$  acts on  $X^k/K_1 \cong X^{k-1}$ , and that  $(K^k/K_1 : UK_1/K_1) = (K^k : UK_1) = \frac{(K^k : U)}{(UK_1 : U)} = p^{m-\ell}$ , applying the induction hypothesis for  $k - 1$  we find that the number of orbits of this action is  $\mathcal{O}((m - \ell)^{k-1})$ . Lemma 13 implies that  $U$  itself has  $\mathcal{O}(\ell(m - \ell)^{k-1}) = \mathcal{O}(m^k)$  orbits, and the proof of the general case is also complete.  $\square$

Lemma 14, Lemma 7 and Theorem 2 together imply that a subgroup  $U < H_k$  of index  $p^n$  which contains  $B$  needs  $\mathcal{O}(n^k)$  generators. Indeed  $U$  can be generated by taking some generating system of  $U/B$ , and one element of  $B$  for each orbit of  $U$ . By Theorem 2 and Lemma 7 we can do the first using  $\mathcal{O}(n)$  elements, and by Lemma 14  $\mathcal{O}(n^k)$  suffices for the second. Thus, the proof of Theorem 4 is complete for the case  $B < U$ . To bound the number of generators of subgroups with non-trivial intersection with  $B$  we have to consider the structure of  $B$  as an  $H_k/B$ -module.

**Lemma 15.** *Let  $G$  be an  $m$ -generated  $p$ -group,  $M$  a  $d$ -generated  $\mathbb{F}_p G$ -module. Let  $N$  be a submodule of  $M$ , which as an  $\mathbb{F}_p$ -vector space has codimension 1. Then  $N$  is a  $(d + m - 1)$ -generated  $\mathbb{F}_p G$ -module.*

*Proof.* Let  $g_1, \dots, g_m$  be generators of  $G$ ,  $v_1, \dots, v_d$  be generators of  $M$  as an  $\mathbb{F}_p G$ -module. Then  $\{v_i^g | i \leq d, g \in G\}$  generates  $M$  as an  $\mathbb{F}_p$ -vector space. Let  $\varphi : M \rightarrow \mathbb{F}_p$  be the module homomorphism given by the canonical map  $M \rightarrow M/N$ . Notice that  $\ker \varphi = N$ . Since the action of a  $p$ -group on a cyclic group of order  $p - 1$  is trivial, we have that  $\mathbb{F}_p$  is a trivial  $\mathbb{F}_p G$ -module, that is, for all  $m \in M$  we obtain  $\varphi(m^g) = \varphi(m)^g = \varphi(m)$ .

Suppose without loss of generality that  $\varphi(v_1) \neq 0$ . We claim that as a vector space  $N$  is generated by

$$D = \{ \varphi(v_i)v_1^g - \varphi(v_1)v_i^h | 1 \leq i \leq d, g, h \in G \}.$$

As  $\varphi(\varphi(v_i)v_1^g - \varphi(v_1)v_i^h) = 0$  we have that  $D \subseteq N$ . Let  $m \in N$  be an arbitrary element. Write  $m = \sum_{i,j} \lambda_{ij} v_i^{h_j}$ , where the  $h_j \in G$ . We have that

$$\sum_{i,j} \lambda_{ij} v_i^{h_j} + \sum_{i,j} \frac{\lambda_{ij}}{\varphi(v_1)} \underbrace{(\varphi(v_i)v_1 - \varphi(v_1)v_i^{h_j})}_{\in D} = \left( \sum_{i,j} \frac{\lambda_{ij}}{\varphi(v_1)} \varphi(v_i) \right) v_1.$$

The left-hand side is in  $N$ , and since  $v_1$  is not in  $N$ , the right-hand side can only be in  $N$  if it vanishes. But then we have represented  $m$  as a linear combination of elements of  $D$ , that is,  $D$  generates  $N$  as a vector space.

We next claim that as an  $\mathbb{F}_p G$ -module  $N$  is generated by the set

$$\{v_1 - v_1^{g^1}, v_1 - v_1^{g^2}, \dots, v_1 - v_1^{g^m}, \varphi(v_i)v_1 - \varphi(v_1)v_2, \dots, \varphi(v_i)v_1 - \varphi(v_1)v_d\}.$$

Let  $V$  be the  $\mathbb{F}_p G$ -module generated by this set. It suffices to show that  $D \subseteq V$ . We show first that  $v_1 - v_1^g \in V$  for all  $g \in G$ . To do so write  $g$  as a word in  $\{g_1^{\pm 1}, \dots, g_m^{\pm 1}\}$ , say  $g = g_{i_1}^{\epsilon_1} \dots g_{i_k}^{\epsilon_k}$ . Then

$$v_1 - v_1^g = v_1 - v_1^{g_{i_k}^{\epsilon_k}} + \left(v_1 - v_1^{g_{i_{k-1}}^{\epsilon_{k-1}}}\right)^{g_{i_k}^{\epsilon_k}} + \dots + \left(v_1 - v_1^{g_{i_1}^{\epsilon_1}}\right)^{g_{i_2}^{\epsilon_2} \dots g_{i_k}^{\epsilon_k}},$$

and since  $v_1 - v_1^{g^{-1}} = -(v_1 - v_1^g)^{g^{-1}}$ , we find that  $v_1 - v_1^g \in V$  for all  $g \in G$ .

Next for  $g, h \in G$  we have  $\varphi(v_i)v_1^g - \varphi(v_1)v_i^h \in V$ , since

$$\varphi(v_i)v_1^g - \varphi(v_1)v_i^h = \varphi(v_i) \left(v_1 - v_1^{hg^{-1}}\right)^g + (\varphi(v_i)v_1 - \varphi(v_1)v_i)^h.$$

We have found a generating system consisting of  $d + m - 1$  elements, and our claim follows.  $\square$

**Lemma 16.** *Let  $U$  be a subgroup of  $H_k$  of index  $p^m$ , and suppose that  $X^k$  decomposes into  $N$  orbits with respect to  $\pi(U)$ . Then the number of generators of  $U$  is at most  $N + \mathcal{O}(m^2)$ .*

*Proof.* Let  $O_1, \dots, O_N$  be a complete list of the orbits of  $\pi(U)$ . For each orbit pick an element  $x_i \in O_i$ , and define  $b_i \in B$ , which its  $x_i$  coordinate is 1 and 0 everywhere else. Then  $b_1, \dots, b_N$  generate  $B$  as a  $\pi(U)$ -module. From Lemma 7 it follows that  $\pi^{-1}(\pi(U)) \geq U$  is generated by  $N + d(\pi(U)) = N + \mathcal{O}(m)$  as  $\pi(U) \leq K^k$  and has index at most  $p^m$  there.

Since  $B$  is abelian we observe that the action of  $U$  on  $B$  factorize via  $\pi(U)$ . Therefore, we need to bound the number of generators  $U \cap B$  as  $\pi(U)$ -module. As  $\pi(U)$  is a  $p$ -group and  $U \cap B$  is of finite index in  $B$  there exists a sequence of  $\pi(U)$ -submodules  $B = M_0 > M_1 > \dots > M_\ell = U \cap B$  with  $(M_j : M_{j+1}) = p$ . We can repeatedly apply Lemma 15 to find that for each  $1 \leq j \leq \ell$  we have that  $M_j$  can be generated by  $\leq N + jd(\pi(U)) = N + \mathcal{O}(jm)$  elements. Hence,  $U$  can be generated by  $\leq N + \mathcal{O}(m^2)$  elements.  $\square$

We have shown that  $H_k$  is a residually finite  $p$ -group, therefore, its subgroup growth is the same as its pro- $p$  completion  $\widehat{H}_k$ . Moreover, the inclusion  $H_k \hookrightarrow \widehat{H}_k$  induces a bijection between finite index subgroups of  $H_k$  and open subgroups of  $\widehat{H}_k$ . Clearly, the number of generator of a subgroup is an upper bound to the number of (topological) generators of its image. Thus,  $d_{H_k}(m) \geq d_{\widehat{H}_k}(m)$ . Combining Lemma 14 and Lemma 16 we obtain that for  $k \geq 2$  we have  $d(m) = \mathcal{O}(m^k)$ . From [6, Proposition 1.6.2] we conclude that  $s_{p^m}(H_k) = s_{p^m}(\widehat{H}_k) = p^{\mathcal{O}(m^{k+1})}$ . On the other hand, from Corollary 12 we have that  $H_k$  has at least  $n^{(\log n)^k}$  subgroup. Thus,  $H_k$  and  $\widehat{H}_k$  have subgroup type  $n^{(\log n)^k}$ . The cases  $k = 0$ , that is,  $p$ -adic analytic groups, and  $k = 1$ , e.g. linear groups over  $\mathbb{F}_p[[t]]$ , have been known before, hence Theorem 4 is true for all  $k$ .

## 5. THE ACTION ON FAMILIES OF SUBTREES

In the previous section we let a product of  $k$  branch groups act on  $k$ -tuples of paths in the  $p$ -adic tree, and obtained groups with subgroup growth type  $n^{\log^k n}$  by

taking wreath products. Here our goal is to construct groups with larger growth by letting  $k$  tend to infinity. This seems impossible as when  $k$  is infinity the group is no longer finitely generated. Nevertheless, branch groups give us the solution as we can increase  $k$  by taking subgroups of finite index. However, we should no longer act on paths, but on subtrees as they can have  $k$  elements in each level and  $k$  can tend to infinity when the level increases.

For a  $p$ -group  $G$  we write  $\Psi(G) = G^p[G, G]$ . Note that in general  $\Psi(G)$  is not the Frattini subgroup, not even if  $G$  is residually finite, however, it always equals the intersection of all finite index maximal subgroups.

**Lemma 17.** *Let  $G$  be a finitely generated residually finite  $p$ -group acting on  $T$  which contains a normal self-replicating subgroup  $K$  of finite index. Let  $\ell$  be the least integer such that  $\Psi(K) \geq K^{p^\ell}$  geometrically. Then  $G$  has the congruence subgroup property, more precisely, if  $U$  is a subgroup of index  $p^m$ , then  $U$  contains  $K^{p^{m\ell}}$  geometrically.*

*Proof.* First note that it suffices to consider the case  $G = K$ . In fact, if  $U < G$  has finite index, then  $(K : U \cap K) \leq (G : U)$ , and we obtain  $U \geq K \cap U \geq K^{p^{m\ell}}$ . Now let  $U$  be a subgroup of  $K$  of finite index  $p^m$ . We prove by induction on  $m$  that  $U$  contains  $K^{p^{m\ell}}$ . If  $m = 1$ , then  $U \geq \Psi(K) \geq K^{p^\ell}$ , and our claim follows. In general  $U$  is contained in a maximal subgroup of index  $p$ , say  $V$ . Since  $V$  is maximal,  $V$  contains  $\Psi(K) \geq K^{p^\ell}$ . Let  $K_i \cong K$  be the  $i$ -th direct factor of  $K^{p^\ell}$ . Then  $(K_i : K_i \cap U) = (K_i U : U) \leq (V : U) = p^{m-1}$ . Thus, from the induction hypothesis each  $K_i \cap U$  contains  $K^{p^{(m-1)\ell}}$  geometrically, and  $U$  contains the direct product of these groups, which is  $K^{p^{m\ell}}$ . Hence,  $U$  is a congruence subgroup of  $K$  of level  $\leq m\ell$ .  $\square$

For  $p = 2$  let  $G$  be the Grigorchuk group and for  $p > 2$  let it be the Gupta-Sidki group, let  $K$  the self-replicating subgroup of  $G$ ,  $St(\ell)$  the stabilizer of the  $\ell$ -th level. Let  $x_1, x_2, \dots, x_n$  be vertices in  $T$ , we write  $G_{x_1 x_2, \dots, x_n}$  for the stabilizer of  $x_1, \dots, x_k$  in  $G$ . We say that a node  $x$  in  $T$  is the sibling of a node  $y \neq x$ , if they have the same parent. If  $S$  is a subtree of  $T$ , which contains the root of  $T$ , and  $x \in T \setminus S$  is a node, we define the splitting point  $y$  of  $S$  with respect to  $x$  to be the unique point on the geodesic path  $p$  from the root to  $x$ , which is contained in  $S \cap p$ , but whose child on  $p$  is not contained in  $S$ . The child of the splitting point, in which the branch containing  $x$  starts, will be called the bud of  $x$  with respect to  $S$ . If  $A$  is a set of nodes, we define the bud of  $x$  with respect to  $A$  to be the bud of  $x$  with respect to the smallest subtree of  $T$  which contains  $A$  and the root of  $T$ . The bud of a node with respect to the empty set is simply the root of the tree.

Given  $u \in T$  let  $T_u$  be the subtree with root  $u$ . The obvious isomorphism of  $T$  and  $T_u$  induces an action of  $G$  on  $T_u$ , we denote by  $\overline{G}_u$  the permutation group defined by this action, and by  $\overline{K}_u$  the image of  $K$  in  $\overline{G}_u$ . If  $H$  is a group of automorphisms of the tree, and  $x_1, \dots, x_n$  are pairwise incomparable points, which are fixed by  $H$ , define  $R_{x_1 \dots x_n}(H)$  to be the restriction of  $H$  to the disjoint union of the trees  $T_{x_1}, \dots, T_{x_n}$ . By the construction of  $G$  we have that  $R_u(G_u) \leq \overline{G}_u$  and as  $K$  is self-replicating  $\overline{K}_u \leq R_u(G_u)$ . From this we obtain the following.

**Lemma 18.** *Suppose  $x_1, \dots, x_k$  is an antichain of vertices in  $T$ . Then we have that*

$$\overline{K}_{x_1} \times \overline{K}_{x_2} \cdots \times \overline{K}_{x_k} \leq R_{x_1 \dots x_k}(G_{x_1 \dots x_k}) \leq \overline{G}_{x_1} \times \overline{G}_{x_2} \cdots \times \overline{G}_{x_k}.$$

From Lemma 17 and the computations by Garrido and Wilson[4] already used in Corollary 3 we obtain

**Corollary 19.** *Let  $G$  be the Grigorchuk group or a Gupta-Sidki group. Then a subgroup of index  $p^m$  contains  $K^{p^{2^m}}$  geometrically.*

The corollary is essentially optimal. In fact, if  $x$  is a point in the  $\ell$ -th level, then  $G_x$  has index  $p^{C^\ell}$  and does not contain the principal congruence subgroup of level  $\ell - 1$ . However, compared to algebraic groups this bound is very weak. The  $\ell$ -th congruence subgroup has index double exponential in  $\ell$ , that is, the largest principal congruence subgroup contained in  $G_x$  has index exponential in the index of  $G_x$ , whereas a congruence subgroup  $U$  in an algebraic group  $G$  usually<sup>1</sup> contains a principal congruence subgroup  $P$  of index polynomial in the index of  $U$ . This difference will cause problems later and is the reason that we cannot at present extend the range of our construction beyond  $n^{(\log n)^{\log^* n}}$ .

Let  $S$  be a rooted subtree of  $T$  with root equal to the root of  $T$ , which has no leafs. Then every element of  $G$  maps  $S$  to an isomorphic copy with the same root. In general  $T$  contains uncountably many isomorphic copies of  $S$ , thus the action of  $G$  on this set cannot be transitive. Pick an orbit  $X$  of  $G$  on the set of rooted isomorphic copies of  $S$ .

**Lemma 20.** *If  $S$  contains no  $T_u$  for any  $u$ , then  $X$  is countably infinite.*

*Proof.* Since  $G$  is countable,  $X$  is at most countable. Since  $S$  has no leafs,  $S$  contains an infinite path  $p = (x_1, x_2, \dots)$ . Suppose that  $X$  is finite, say  $X = \{g_1 S, g_2 S, \dots, g_n S\}$ ,  $g_i S \neq g_j S$ . Then there is some level  $\ell$ , such that each of the finitely many elements  $g_1, \dots, g_n$  is uniquely determined by its action on the  $\ell$ -th level. In particular if  $g \in St(\ell)$ , then  $gg_i S = g_i S$ . Pick a point  $x$  on the  $\ell$ -th level, which is contained in  $g_i S$ . Since  $S$  does not contain  $T_x$ , we can pick some  $y \in T_x \setminus g_i S$  of level  $\ell' > \ell$ . Since  $S$  contains no leafs,  $g_i S$  contains an infinite path passing through  $x$ , in particular, there is a point  $z \in g_i S$  of level  $\ell'$ . Since  $St(\ell)$  acts transitively on  $T_x$ , and  $y, z \in T_x$  are on the same level, there is some  $g \in St(\ell)$ , which maps  $z$  to  $y$ . But then  $gg_i S \neq g_i S$ , since  $y \in gg_i S$  and  $y \notin g_i S$ , while  $gg_i S = g_i S$  since  $g$  is in the  $\ell$ -th congruence subgroup. This contradiction implies our claim.  $\square$

The following corresponds to Lemma 14.

**Lemma 21.** *There exists a constant  $C$  depending only on  $p$  such that if  $U$  is a subgroup of  $G$  of index  $p^m$  that contains  $St(\ell)$ , and  $k$  is the number of points of  $S$  of level  $\ell$ , then under the  $U$ -action  $X$  decomposes into at most  $(Cm)^k$  orbits.*

*Proof.* Let  $U$  be a subgroup of index  $p^m$  that contains  $St(\ell)$ . Let  $L$  be the  $\ell$ -th level of the tree, and put  $\{x_1, \dots, x_k\} = L \cap S$ . Let  $y_i$  be the bud of  $x_i$  with respect to  $x_1, \dots, x_{i-1}$ , and let  $\ell_i$  be the level of  $y_i$ . Put  $X_L = \{S^g \cap L : g \in G\}$ . We first count the number of orbits of  $U$  on  $X_L$ , then count the number of orbits of  $U$  on  $X$ , which correspond to a single orbit in  $X_L$ .

The one-dimensional case of Lemma 14 implies that the orbit of  $x_1$  under  $G$ , that is, all of  $L$ , decomposes into at most  $C_1 m$  orbits under  $U$ .

Now  $U_{x_1}$  fixes the geodesic path between the root and  $x_1$ , in particular  $U_{x_1}$  fixes the parent of  $y_2$ . Both the parent and a sibling of  $y_2$  are fixed by  $U_{x_1}$ , hence  $y_2$

<sup>1</sup>For a precise statement see [6, Proposition 6.1.1].

itself is fixed by  $U_{x_1}$ , as a  $p$ -group acting on  $p-1$  points must act trivially. Hence,  $U_{x_1}$  acts on  $T_{y_2}$  and  $R_{y_2}(U_{x_1})$  is defined. By Lemma 15 we have that  $R_{y_2}(G_{x_1})$  contains  $\overline{K}_{y_2}$ . Put  $V = \overline{K}_{y_2} \cap R_{y_2}(U_{x_1})$ . We would like to apply Lemma 14 again to  $U_{x_1}$  acting on  $T_{y_2}$ . To do so we have to bound  $(\overline{G}_{y_2} : R_{y_2}(U_{x_1}))$ . We have

$$(\overline{G}_{y_2} : R_{y_2}(U_{x_1})) \leq (\overline{G}_{y_2} : V) \leq (\overline{G}_{y_2} : \overline{K}_{y_2})(\overline{K}_{y_2} : V) \leq (G : K)(G : U),$$

since

$$(G : U) \geq (G_{x_1} : U_{x_1}) \geq (R_{y_2}(G_{x_1}) : R_{y_2}(U_{x_1})) \geq (\overline{K}_{y_2} : V).$$

Therefore,  $R_{y_2}(U_{x_1})$  is a subgroup of  $\overline{G}_{y_2}$  of index at most  $p^{m+c}$  for some  $c$  depending only on  $p$ . Using the one-dimensional case of Lemma 14 regarding  $T_{y_2}$  we find that  $R_{y_2}(U_{x_1})$  has at most  $C_1(m+c) \leq C_2m$  orbits on the set of points of level  $\ell$  below  $y_2$ . We deduce that  $U$  has at most  $(C_2m)^2$  orbits on the set  $(x_1, x_2)^G$ . These pairs are exactly the pairs of points that arise by intersecting the first two paths of a tree in  $X$  with  $L$ . Continuing in this way we find that  $U$  has  $(C_2m)^k$  orbits on  $X_L$ .

Now we consider a single orbit  $\Omega$  of  $U$  on  $X_L$ . For a fixed tuple  $\vec{z} = (z_1, \dots, z_k) \in \Omega$  consider the set  $X_{\vec{z}} = \{R \in X : R \cap L = \vec{z}\}$ . We have to count the number of orbits of  $U$  on  $\bigcup_{\vec{z} \in \Omega} X_{\vec{z}}$ . Each orbit of  $U$  on  $\bigcup_{\vec{z} \in \Omega} X_{\vec{z}}$  intersects  $X_{\vec{z}}$ , and the intersection is an orbit of  $U_{\vec{z}}$ . Hence we have a bijection between the orbits of  $U$  on  $\bigcup_{\vec{z} \in \Omega} X_{\vec{z}}$  and the orbits of  $U_{\vec{z}}$  on  $X_{\vec{z}}$ . To bound the number of the latter note that by assumption  $St(\ell) \leq U_{\vec{z}}$ , thus  $K^{p^\ell} \leq U_{\vec{z}}$ . Hence the number of orbits of  $U_{\vec{z}}$  on  $X_{\vec{z}}$  is at most equal to the number of orbits of  $K^{p^\ell}$  on  $X_{\vec{z}}$ . Clearly the number of orbits does not change if we replace  $K^{p^\ell}$  by  $\overline{K}_{z_1} \times \dots \times \overline{K}_{z_k}$ . Since  $\overline{G}_{z_1} \times \dots \times \overline{G}_{z_k}$  acts transitively on  $X_{\vec{z}}$ , we have that the number of orbits of  $\overline{K}_{z_1} \times \dots \times \overline{K}_{z_k}$  on this set is at most

$$(\overline{G}_{z_1} \times \dots \times \overline{G}_{z_k} : \overline{K}_{z_1} \times \dots \times \overline{K}_{z_k}) = (G : K)^k.$$

Thus the total number of orbits is bounded by  $(C_2(G : K)m)^k$ , and the statement of the lemma follows with  $C = C_2(G : K)$ .  $\square$

In the special case  $U = St(\ell)$  we can be more precise, in fact, the same argument yields a lower bound as well, which is analogous to the first part of the proof of Lemma 11. We consider the  $p$ -adic tree not as a graph, but as a graph together with a cyclic ordering of the children of a node. In particular we require that an isomorphism of a subtree also respects this ordering. Note that as  $G$  is a  $p$ -group, which for each node  $x$  contains an element acting as a cyclic shift on the children of  $x$ , all elements of  $G$  preserve the cyclic order.

**Lemma 22.** *Let  $L$  be the  $\ell$ -th level of the tree, and put  $\{x_1, \dots, x_k\} = L \cap S$ . Denote by  $p_i$  the geodesic path from the root to  $x_i$ . Let  $y_i$  be the bud of  $x_i$  with respect to  $x_1, \dots, x_{i-1}$ . Let  $\ell_i$  be the level of  $y_i$ . Then the number of orbits of  $St(\ell)$  on  $X$  is at least*

$$\frac{1}{|\text{Aut}(\overline{S})|} p^{(\sum_{i=1}^k \ell - \ell_i)},$$

where  $\overline{S}$  is the intersection of  $S$  with the first  $\ell$  levels of  $T$ .

*Proof.* Every isomorphic embedding of  $\overline{S}$  into the  $p$ -adic tree of height  $\ell$  is induced by some element of  $G$ . On the other hand  $St(\ell)$  acts trivially on the set of embedded copies of  $\overline{S}$ . Therefore the number of orbits of  $St(\ell)$  on  $X$  is at least as large as the

number of embedded copies of  $\overline{S}$ . This number equals the number of embeddings divided by the number of automorphisms of  $\overline{S}$ . To count the number of embeddings we first map  $p_1$  to an arbitrary path. This can be done in  $p^\ell = p^{\ell-\ell_1}$  ways, as  $y_1$  is the root of the tree, and therefore  $\ell_1 = 0$ . Next the path  $p_2$  is determined up to the parent of  $y_2$  as it is a part of  $p_1$ , and  $y_2$  itself is determined as one of its siblings is contained in  $p_1$  and therefore determined. Below  $y_1$  there are no further restrictions, hence the number of possibilities for  $p_2$  equals  $p^{\ell-\ell_2}$ . Continuing in this way we find that the total number of possibilities equals  $p^{\sum_{i=1}^k \ell - \ell_i}$ , and our claim follows.  $\square$

## 6. CONSTRUCTING GROWTH TYPES USING A SPECIAL CLASS OF TREES

Lemma 21 has two weaknesses. First, the bound becomes much too large, if the tree  $S$  has a large automorphism group, or if  $S \cap L$  contains many pairs  $x, y$  of nodes with small distance. In fact, in the latter case the orbit of  $x$  under  $G_y$  is small, thus deleting all paths through  $x$  from  $S$  reduces the number of orbits by  $(G_y : G_{xy})$  at most, which can be much smaller than the bound  $\mathcal{O}(m)$  implied by Lemma 14. This problem could be overcome by considering special trees.

Second, for our application to subgroup growth we want to use Lemma 21 for an upper bound, and Lemma 22 for a lower bound. This only works well if arbitrary groups behave similar to principal congruence subgroups. Unfortunately for an arbitrary subgroup  $U$ , Corollary 19 only yields the existence of a principal congruence subgroup  $C_\ell$  in  $U$  with index  $(G : C_\ell)$  exponential in  $(G : U)$ . This problem could only be solved by generalizing the notion of a principal congruence subgroup by considering stabilizers of arbitrary maximal anti-chains in the tree. These subgroups would no longer be normal, thus the strategy of proof of Lemma 17 breaks down. Worse, the analogue of Lemma 22 would involve the relative position of the paths already chosen to the anti-chain defining the subgroup, thus even formulating a reasonable conjecture appears difficult.

While we believe that understanding stabilizers of maximal anti-chains could give useful insight into branch groups, this is not within the scope of this paper. Instead we focus on trees, for which Lemma 21 gives good results. We want trees, which have no automorphisms, for which  $S \cap L$  mostly contain points which are far apart, and, if  $|\ell - \ell'|$  is small, then  $S \cap L$ ,  $S \cap L'$  look very similar, if  $L, L'$  are level sets to level  $\ell, \ell'$ , respectively.

We consider trees  $S$ , such that all splitting points lie on a single path  $p$ , which we call the stem of  $S$ . Moreover, we require that each splitting point has exactly two children. For these trees the only relevant information needed to apply Lemma 21 and 22 is the level of the splitting points. We denote by  $a_1, a_2, \dots$  the levels of the buds of  $S$ , and put  $f(t) = \max\{k : a_k \leq t\}$ , that is, the number of buds of level at most  $t$ . As in the previous section let  $X$  be the orbit of  $S$  under the action of  $G$ . For a finite index subgroup  $U$  of  $G$  define  $\lambda(U)$  to be the number of orbits of  $U$  acting on  $X$ . Let  $H_f = G \wr \mathbb{F}_p$  be the wreath product with respect to the action of  $G$  on  $X$ . As in section 4 we see that Lemma 9 implies that  $H_f$  is residually finite.

**Lemma 23.** *There exists a constant  $C$  such that for a subgroup  $U < G$  of index  $p^m$  containing  $St(\ell)$  we have  $\lambda(U) \leq (Cm)^{f(\ell)+1}$ , and for every  $\epsilon \in (0, 1)$  we have  $\lambda(St(\ell)) \geq p^{(1-\epsilon)f(\ell)\ell}$ .*

*Proof.* The upper bound follows immediately from Lemma 21. For the lower bound we compute the bound in Lemma 22. The intersection  $\overline{S}$  of  $S$  with the first  $\ell$  levels has  $f(\ell) + 1$  leaves. For each  $1 \leq i \leq f(\ell) - 1$  there is a unique leaf in  $S$ , such that the path from the root to this leaf contains exactly  $k$  splitting points, and there are two leaves for which this path contains  $f(\ell)$  splitting points. Hence,  $|\text{Aut}(\overline{S})| \leq 2$ . Thus, the number of orbits is bounded below by

$$\frac{1}{|\text{Aut}(\overline{S})|} p^{\ell + \sum_{i=1}^{f(\ell)} \ell - \ell_i} \geq \frac{1}{2} p^{\ell + \sum_{i=1}^{f(\ell)} \ell - a_i} \geq p^{\sum_{i=1}^{f(\ell)} \ell - a_i} \geq p^{(1-\epsilon)f(\ell)\ell},$$

where for the last inequality we neglected all branches which do not bud within the first  $\epsilon\ell$  levels.  $\square$

**Lemma 24.** *Define  $f$  and  $H_f$  as above, and assume that  $f \nearrow \infty$ . Then  $\widehat{H}_f$ , the pro- $p$  completion of  $H_f$ , satisfies the following:*

- (1) *For every  $\epsilon \in (0, 1)$  and  $m > p^2$  we have  $d(m) \geq m^{(1-\epsilon-\frac{2\log p}{\log m})f(\frac{\epsilon \log m}{2\log p})}$  and  $s_{p^m}(\widehat{H}_f) \geq p^{(m-1)^{(1-\epsilon-\frac{\log p}{\log m})f(\frac{\epsilon \log m}{3\log p})-1}$ .*
- (2) *There exists a constant  $C$ , such that for  $d(m) \leq (Cm)^{f(Cm)+1}$  and  $s_{p^m}(\widehat{H}_f) \leq p^{(Cm)^{f(Cm)+2}}$ .*

*Proof.* We begin with the lower bound for  $d(m)$ . For an integer  $m$  set  $\ell = \left\lfloor \frac{\log m}{\log p} \right\rfloor$ . By Lemma 10 we have  $(G : St(\ell)) \leq p^m$ , pick a subgroup  $\overline{U}$  of  $G$  of index  $p^m$  with  $\overline{U} \leq St(\ell)$ . Clearly, the number of  $\overline{U}$ -orbits on  $X$  is at least equal to the number of  $St(\ell)$ -orbits on  $X$ , which by Lemma 23 is at least  $p^{(1-\epsilon)f(\ell)\ell}$ . Let  $U \leq H_f$  be the preimage of  $\overline{U}$  under the canonical projection  $H_f \rightarrow G$ . As in the proof of Lemma 11 we deduce that  $U$  maps surjectively onto  $\mathbb{F}_p^d$ , where  $d \geq p^{(1-\epsilon)f(\ell)\ell}$ . Since  $H_f$  is a  $p$ -group, the kernel of  $U$  is a normal subgroup of index a power of  $p$ , thus the closure of  $U$  in the pro- $p$  completion of  $H_f$  needs at least  $p^{(1-\epsilon)f(\ell)\ell}$  generators as a pro- $p$  group. By our assumption  $m \geq p^2$  we have  $\ell \geq \frac{\log m}{\log p} - 1 \geq \frac{\log m}{2\log p}$ , thus

$$d(m) \geq p^{(1-\epsilon)f(\frac{\epsilon \log m}{2\log p})(\frac{\log m}{\log p} - 1)} \geq p^{(1-\epsilon-\frac{\log p}{\log m})f(\frac{\epsilon \log m}{2\log p})\frac{\log m}{\log p}} = m^{(1-\epsilon-\frac{\log p}{\log m})f(\frac{\epsilon \log m}{2\log p})}.$$

For the upper bound we do not have to distinguish between  $H_f$  and  $\widehat{H}_f$ , as a generating set of a subgroup  $U$  of  $H_f$  is automatically a topological generating set of the topological closure of  $U$  in  $\widehat{H}_f$ . Now let  $U$  be a subgroup of  $H_f$  of index  $p^m$ ,  $\overline{U}$  be the image under the projection  $H_f \rightarrow G$ . By Corollary 19 there is some  $C_1$  such that  $\overline{U}$  contains  $St(C_1 m)$ . Hence from Lemma 23 we see that  $\overline{U}$  has at most  $(C_2 m)^{f(C_1 m)+1}$  orbits on  $X$ . As in the proof of Lemma 14 we first deduce that  $UB$  can be generated by  $(C_2 m)^{f(C_1 m)+1}$  elements, where  $B$  is the base group of the wreath product, and then we apply Lemma 15 to see that  $U$  itself can be generated by  $(C_2 m)^{f(C_1 m)+1} + C_3 m^2 \leq (2C_2 m)^{f(C_1 m)+1}$  elements, provided that  $m$  is sufficiently large. Taking  $C = C_1 + 2C_2$ , the upper bound for  $d(m)$  now follows.

From Proposition 1 (3) and the bounds for  $d(m)$  we obtain

$$\begin{aligned} p^{(m-1)^{(1-\epsilon-\frac{2\log p}{\log m})f(\frac{\epsilon \log m}{3\log p})-1}} &\leq p^{(m-1)^{(1-\epsilon-\frac{\log p}{\log(m-1)})f(\frac{\epsilon \log(m-1)}{2\log p})-1}} \leq p^{d(m-1)-1} \\ &\leq s_{p^m}(\widehat{H}_f) \leq p^{\sum_{\mu=1}^{m-1} d(\mu)} \leq p^{\sum_{\mu=1}^{m-1} (C\mu)^{f(C\mu)+1}} \leq p^{(Cm)^{f(Cm)+2}}, \end{aligned}$$

as claimed.  $\square$

If  $f(t) > \frac{t}{\log t}$ , then the upper bound for  $s_{p^m}(\widehat{H}_f)$  becomes worse than the trivial bound obtained by comparing  $\widehat{H}_f$  with the free pro- $p$  group. On the other hand for  $f(t) < \frac{t}{\log t}$  the lower bound is smaller than  $p^{m^{\log m}}$ , which is of type  $e^{(\log n)^{\log \log n}}$ . Thus if we try to construct groups of growth type larger than  $e^{(\log n)^{\log \log n}}$  using this method, the upper bound implied by Lemma 24 becomes trivial. However, for slowly growing functions we obtain the following.

**Corollary 25.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function, tending to infinity, which satisfies  $f(p^t) \leq f(t) + 1$  for all  $t \in \mathbb{N}$ . Then there exists a pro- $p$  group  $\widehat{H}_f$ , such that  $s_{p^m}(\widehat{H}_f) = p^{m^{f(m)+\mathcal{O}(1)}}$ , that is,  $\widehat{H}_f$  has growth type  $n^{(\log n)^{f(m)+\mathcal{O}(1)}}$ .*

*Proof.* Our assumptions imply  $f(m) = o(\log m)$ . From Lemma 24 we have for  $m$  sufficiently large

$$\frac{\log s_{p^m}(\widehat{H}_f)}{\log p} \leq (Cm)^{f(Cm)+2} \leq \underbrace{C^{f(p^m)+2}}_{< m} m^{f(p^m)+2} \leq m^{f(m)+4} = m^{f(m)+\mathcal{O}(1)}.$$

For the lower bound we set  $\epsilon = \frac{1}{\log \log m}$  and obtain for  $m$  sufficiently large

$$\frac{\log s_{p^m}(\widehat{H}_f)}{\log p} \geq (m-1)^{(1-\epsilon-\frac{2 \log p}{\log m})f(\frac{\epsilon \log m}{3 \log p})} \geq (m-1)^{f(\log \log m) - (\frac{1}{\log \log m} + \frac{2 \log p}{\log m})f(\log m)},$$

since  $\log \log m \leq \frac{\epsilon \log m}{3 \log p} \leq \log m$ . As  $f(m) = o(\log m)$  we have

$$\left( \frac{1}{\log \log m} + \frac{2 \log p}{\log m} \right) f(\log m) = o(1).$$

Using  $f(\log \log m) \geq f(m) - 2$  and Bernoulli's inequality  $(1+x)^r \geq 1+rx$ , we get

$$\begin{aligned} \frac{\log s_{p^m}(\widehat{H}_f)}{\log p} &\geq (m-1)^{f(\log \log m)-1} \geq (m-1)^{f(m)-3} \geq m^{f(m)-3} \left( \frac{m-1}{m} \right)^{f(m)} \\ &\geq m^{f(m)-3} \left( 1 - \frac{f(m)}{m} \right) \geq m^{f(m)-4} = m^{f(m)+\mathcal{O}(1)}, \end{aligned}$$

and our claim follows.  $\square$

Theorem 5 follows, since the case in which  $f$  does not tend to infinity is covered by Theorem 4.

To see that we have actually constructed many growth types, we have to distinguish growth types for different functions  $f, g$ . Recall that  $f$  is of strict type  $g$ , if there is a constant  $C$ , such that  $f(n) \leq g(n)^C$  and  $g(n) \leq f(n)^C$  holds for all large  $n$ . If  $f(n) \leq g(n)^C$  for all large  $n$ , we say that  $g$  dominates  $f$ . If  $g$  dominates  $f$ , and  $f$  and  $g$  are not of the same strict type, we say that  $g$  strictly dominates  $f$ .

If  $G, H$  are pro- $p$  groups, we say that  $G$  and  $H$  have the same strict type, if  $s_{p^m}(G)$  is of strict type  $s_{p^m}(H)$ , similarly we define domination and strict domination.

**Lemma 26.** *Let  $f, g$  be functions satisfying the conditions of Corollary 25.*

- (1) *If  $\widehat{H}_f$  and  $\widehat{H}_g$  have the same strict type, then  $f - g$  is bounded.*
- (2) *If  $\limsup(f(n) - g(n)) = \infty$ , then  $\widehat{H}_g$  does not dominate  $\widehat{H}_f$ .*
- (3) *If  $\liminf(f(n) - g(n)) = \infty$ , then  $\widehat{H}_f$  strictly dominates  $\widehat{H}_g$ .*



- (4) If  $\limsup(f(n) - g(n)) = \infty$ ,  $\liminf(f(n) - g(n)) = -\infty$ , then the growth types of  $\widehat{H}_f$  and  $\widehat{H}_g$  are incomparable.

*Proof.* If there is a constant  $C$ , such that  $s_{p^m}(\widehat{H}_f) \leq s_{p^m}(\widehat{H}_g)^C$ , then  $m^{f(m)-C_1} \leq C m^{g(m)+C_1}$ , where  $C_1$  is the constant implied by the Landau symbol in Lemma 24. This implies that  $f(m) \leq g(m) + 2C_1 + 1$  for all large  $m$ , and (2) follows. (1) and (4) follow by applying (2) twice. For (3) notice that for  $m$  large we have  $s_{p^m}(\widehat{H}_f) \geq p^{m^{f(m)-C_1}}$ ,  $s_{p^m}(\widehat{H}_g) \leq p^{m^{g(m)+C_1}}$ , hence, if  $\liminf f(n) - g(n) = \infty$ , then  $s_{p^m}(\widehat{H}_f) \geq s_{p^m}(\widehat{H}_g)$  for all large  $m$ , that is,  $\widehat{H}_f$  dominates  $\widehat{H}_g$ , while (1) implies that  $\widehat{H}_f$  and  $\widehat{H}_g$  are not of the same strict type.  $\square$

Recall that  $\log^* n$  is the least  $k$  such that the exponential tower  $p^{p^{\cdot^{\cdot^p}}}$  of height  $k$  supersedes  $n$ . For  $\alpha \in (0, 1)$  define  $f_\alpha(n) = \lfloor (\log^* n)^\alpha \rfloor$ . Then  $f_\alpha$  satisfies the condition of Corollary 25, and for  $\alpha > \beta$  we have  $(f_\alpha(n) - f_\beta(n)) \rightarrow \infty$ . Hence, from Lemma 26 (3) we find continuum many groups  $\widehat{H}_{f_\alpha}$  which are totally ordered with respect to domination, in particular, there are continuum many different growth types of pro- $p$  groups and the proof of Theorem 5 is finished.

By refining this construction we prove below Corollary 6. As lattices we have that  $((0, 1), <)^\mathbb{N} \cong (\mathbb{R}, <)^\mathbb{N}$ . For each tuple  $\vec{\alpha} = (\alpha_i)_{i \in \mathbb{N}}$ , with  $\alpha_i \in (0, 1)$ , we construct an unbounded non-decreasing function  $f_{\vec{\alpha}} : \mathbb{N} \rightarrow \mathbb{N}$  satisfying  $f_{\vec{\alpha}}(p^t) \leq f_{\vec{\alpha}}(t) + 1$  for all  $t \in \mathbb{N}$ . To obtain an embedding of lattices we have to ensure that:

- (1) For all tuples  $\vec{\alpha}, \vec{\beta}$  with  $\alpha_i \geq \beta_i$  for all indices  $i$ , and  $\alpha_j > \beta_j$  for some index  $j$ , we have that  $\liminf f_{\vec{\alpha}}(n) - f_{\vec{\beta}}(n) = \infty$ ;
- (2) For all tuples  $\vec{\alpha}, \vec{\beta}$  with  $\alpha_j > \beta_j$  for some index  $j$ , we have that  $\limsup f_{\vec{\alpha}}(n) - f_{\vec{\beta}}(n) = \infty$ .

We do so in a couple of steps. We first construct unbounded non-decreasing functions  $g_{\vec{\alpha}} : \mathbb{N} \rightarrow \mathbb{N}$  satisfying (1), (2), and  $g_{\vec{\alpha}}(n) \leq g_{\vec{\alpha}}(n-1) + 1$  for all  $n \in \mathbb{N}$ . We then stretch them by substituting  $\log^*$  to obtain an unbounded non-decreasing functions  $f_{\vec{\alpha}} : \mathbb{N} \rightarrow \mathbb{N}$  satisfying (1), (2), and  $f_{\vec{\alpha}}(p^t) \leq f_{\vec{\alpha}}(t) + 1$ .

Let  $(d_i)$  be a fixed sequence of positive integers, such that each integer occurs infinitely often. For  $\vec{\alpha} = (\alpha_i)$  a sequences of real numbers in  $(0, 1)$  put  $\alpha = \sum \frac{\alpha_i}{2^i}$ . We define  $g_{\vec{\alpha}} : \mathbb{N} \rightarrow \mathbb{N}$  the following way:

$$g_{\vec{\alpha}}(n) = \begin{cases} 2^k + \left\lfloor \frac{(n-2^{2k})^{\alpha_{d_k}} + \alpha \log n}{2} \right\rfloor & 2^{2k} \leq n < 2^{2k} + 2^k, \\ \min(g_{\vec{\alpha}}(n-1) + 1, g_{\vec{\alpha}}(2^{2k+2})) & 2^{2k} + 2^k \leq n < 2^{2k+2}. \end{cases}$$

First let us see that  $g_{\vec{\alpha}}(n) \leq g_{\vec{\alpha}}(n-1) + 1$ . If  $2^{2k} + 2^k \leq n < 2^{2k+2}$ , then this follows immediately from the definition. If  $2^{2k} < n < 2^{2k} + 2^k$ , then

$$g_{\vec{\alpha}}(n) - g_{\vec{\alpha}}(n-1) = \left\lfloor \frac{(n-2^{2k})^{\alpha_{d_k}} + \alpha \log n}{2} \right\rfloor - \left\lfloor \frac{(n-1-2^{2k})^{\alpha_{d_k}} + \alpha \log(n-1)}{2} \right\rfloor.$$

Because  $\alpha_{d_k} < 1$  and  $\alpha < 1$  we have that

$$\frac{(n-2^{2k})^{\alpha_{d_k}} + \alpha \log n}{2} - \frac{(n-1-2^{2k})^{\alpha_{d_k}} + \alpha \log(n-1)}{2} \leq 1$$

and thus,  $g_{\vec{\alpha}}(n) - g_{\vec{\alpha}}(n-1) \leq 1$ . We are left with the case  $n = 2^{2k}$ . Notice that

$$g_{\vec{\alpha}}(2^{2k}) - g_{\vec{\alpha}}(2^{2(k-1)}) = 2^k + \left\lfloor \frac{\alpha k}{2} \right\rfloor - 2^{k-1} - \left\lfloor \frac{\alpha(k-1)}{2} \right\rfloor \leq 2^{2k} - 2^{2(k-1)} - 2^{k-1}.$$

As  $g(m) - g(m-1) = 1$  for  $2^{2(k-1)} + 2^{k-1} \leq m < 2^{2k}$  if  $g_{\bar{\alpha}}(m) \neq g_{\bar{\alpha}}(2^{2k})$  we obtain that  $g_{\bar{\alpha}}(2^{2k} - 1) = g_{\bar{\alpha}}(2^{2k})$ .

It is clear that  $g_{\bar{\alpha}}(n-1) \leq g_{\bar{\alpha}}(n)$  for all  $n$  except possibly for  $n = 2^{2k} + 2^k$ . Since  $g_{\bar{\alpha}}(n) \leq g_{\bar{\alpha}}(n-1) + 1$  we have that  $g_{\bar{\alpha}}(2^{2k} + 2^k - 1) \leq g_{\bar{\alpha}}(2^{2k}) + 2^k - 1 \leq g_{\bar{\alpha}}(2^{2k+2})$ . We deduce that  $g_{\bar{\alpha}}(2^{2k} + 2^k - 1) \leq g_{\bar{\alpha}}(2^{2k} + 2^k)$ . Hence,  $g_{\bar{\alpha}}$  is a non-decreasing function that is obviously unbounded.

Suppose that  $\alpha_i \geq \beta_i$  for all  $i$ , and  $\alpha_j > \beta_j$  for at least one  $j$ , then  $\alpha = \sum \frac{\alpha_i}{2^i} > \sum \frac{\beta_i}{2^i} = \beta$ . Therefore, for  $2^{2k} \leq n < 2^{2k} + 2^k$  we have that

$$\begin{aligned} g_{\bar{\alpha}}(n) - g_{\bar{\beta}}(n) &= \left\lfloor \frac{(n - 2^{2k})^{\alpha_{d_k}} + \alpha \log n}{2} \right\rfloor - \left\lfloor \frac{(n - 2^{2k})^{\beta_{d_k}} + \beta \log n}{2} \right\rfloor \\ &\geq \frac{(n - 2^{2k})^{\alpha_{d_k}} + \alpha \log n}{2} - \frac{(n - 2^{2k})^{\beta_{d_k}} + \beta \log n}{2} - 1 \\ &\geq (\alpha - \beta) \frac{\log n}{2} - 1. \end{aligned}$$

For  $2^{2k} + 2^k \leq n < 2^{2k+2}$  we claim that  $g_{\bar{\alpha}}(n) - g_{\bar{\beta}}(n) \geq (\alpha - \beta)k - 1$ . If  $g_{\bar{\alpha}}(n) = g_{\bar{\alpha}}(2^{2k+2})$ , then  $g_{\bar{\alpha}}(n) - g_{\bar{\beta}}(n) = g_{\bar{\alpha}}(2^{2k+2}) - g_{\bar{\beta}}(n) \geq g_{\bar{\alpha}}(2^{2k+2}) - g_{\bar{\beta}}(2^{2k+2}) \geq (\alpha - \beta)(k+1) - 1 \geq (\alpha - \beta)k - 1$ . Otherwise,  $g_{\bar{\alpha}}(n) \neq g_{\bar{\alpha}}(2^{2k+2})$  which means that  $g_{\bar{\alpha}}(m) - g_{\bar{\alpha}}(m-1) = 1$  for  $2^{2k} + 2^k \leq m < n$  and  $g_{\bar{\beta}}(m) - g_{\bar{\beta}}(m-1) \leq 1$  for  $2^{2k} + 2^k \leq m < n$ . Thus, it suffices to prove the claim for  $n = 2^{2k} + 2^k - 1$ . Indeed,  $g_{\bar{\alpha}}(2^{2k} + 2^k - 1) - g_{\bar{\beta}}(2^{2k} + 2^k - 1) \geq (\alpha - \beta) \frac{\log(2^{2k} + 2^k - 1)}{2} - 1 \geq (\alpha - \beta)k - 1$ . As  $(\alpha - \beta)k - 1$  tends to infinity when  $n$  tends to infinity we deduce that  $g_{\bar{\alpha}}$  and  $g_{\bar{\beta}}$  satisfy property (1).

If  $\alpha_j > \beta_j$  for some  $j$ , then for each  $k$ , such that  $d_k = j$ , we have

$$\begin{aligned} g_{\bar{\alpha}}(2^{2k} + 2^k - 1) - g_{\bar{\beta}}(2^{2k} + 2^k - 1) &= \left\lfloor \frac{(2^k - 1)^{\alpha_j} + \alpha \log(2^{2k} + 2^k - 1)}{2} \right\rfloor \\ &\quad - \left\lfloor \frac{(2^k - 1)^{\beta_j} + \beta \log(2^{2k} + 2^k - 1)}{2} \right\rfloor \\ &\geq \frac{1}{2} ((2^k - 1)^{\alpha_j} - (2^k - 1)^{\beta_j} - \beta(k+1)), \end{aligned}$$

which is unbounded. Therefore,  $g_{\bar{\alpha}}$  and  $g_{\bar{\beta}}$  satisfy property (2).

We now define  $f_{\bar{\alpha}}(n) = g_{\bar{\alpha}}(\lfloor \log^* n \rfloor)$ . Since  $g_{\bar{\alpha}}$  and  $\lfloor \log^* n \rfloor$  are unbounded non-decreasing functions so is  $f_{\bar{\alpha}}$ . Furthermore, as  $g_{\bar{\alpha}}$  satisfies property (1) so is  $f_{\bar{\alpha}}$ . In addition, because  $g_{\bar{\alpha}}$  satisfies property (2) and  $\lfloor \log^* n \rfloor$  is surjective we have that  $f_{\bar{\alpha}}$  satisfies property (2). Finally,

$$f_{\bar{\alpha}}(p^n) = g_{\bar{\alpha}}(\lfloor \log^* p^n \rfloor) = g_{\bar{\alpha}}(\lfloor \log^* n \rfloor + 1) \leq g_{\bar{\alpha}}(\lfloor \log^* n \rfloor) + 1 = f_{\bar{\alpha}}(n) + 1.$$

We conclude that the lattice  $(0, 1)^{\mathbb{N}}$  embeds into the lattice of growth types of pro- $p$  groups.

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